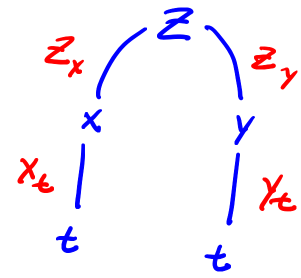


Sec 14.5 CHAIN RULE

Chain Rule Formula. If $z = f(x, y)$ and $x \equiv x(t)$ and $y \equiv y(t)$, then

$$\frac{dz}{dt} = f_x(x(t), y(t))\; x'(t) + f_y(x(t), y(t))\; y'(t)$$

Short Notation:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Ex1. If $z = x^2y + xy^2$, $x = 2 + t^4$, $y = 1 - t^2$, find $\left.\frac{dz}{dt}\right|_{t=1}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\boxed{\frac{dz}{dt} = (2xy + y^2) \cdot (4t^3) + (x^2 + 2xy) \cdot (-2t)}$$

when $t = 1$, $(x, y) = (3, 0)$

now $\left.\frac{dz}{dt}\right|_{t=1} = (2(3) + 0^2) \cdot (4(1)^3) + (9 + 2(3)(0))(-2(1)) = \boxed{-18}$.

In three variables: If $w = f(x, y, z)$ and $x \equiv x(t)$, $y \equiv y(t)$ and $z \equiv z(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Ex2. If $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$ and $z = 1 - 2t$, find $\left.\frac{dw}{dt}\right|_{t=2}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

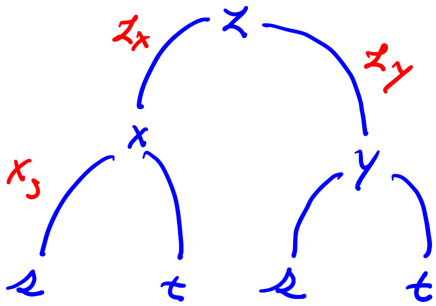
$$\frac{dw}{dt} = (e^{y/z})(2t) + \left(\frac{x}{z} e^{y/z}\right)(-1) + \left(\frac{-xy}{z^2} e^{y/z}\right) \cdot (-2)$$

when $t = 2$, $(x, y, z) = (4, -1, -3)$

then $\left.\frac{dw}{dt}\right|_{t=2} = (e^{1/3})(4) + \left(\frac{4}{3}\right)(e^{1/3}) + \left(\frac{-8}{9}\right) e^{1/3} = e^{1/3}\left(4 + \frac{4}{3} - \frac{8}{9}\right)$

$$= e^{1/3}\left(\frac{40}{9}\right)$$

More general: What if $z = f(x, y)$ and $x \equiv x(s, t)$ and $y \equiv y(s, t)$?
In this case, the composite function is $z = f(x(s, t), y(s, t))$.



Following the diagram, we have the formulas:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \text{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex3. If $z = x^2 + xy + y^2$, $x = 4s + t$ and $y = s^2 t$, compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ at the point $(s, t) = (1, 2)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (2x+y)(4) + (x+2y)\cdot(2st)$$

when $(s,t)=(1,2)$, we get $(x,y)=(6,2)$

then

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(1,2)} = (2(6)+2)(4) + (6+2(2))(2(1)(2)) = \boxed{96}$$

todo: $\left.\dfrac{\partial z}{\partial t}\right|_{\substack{s=1\\t=2}} = ?$

Exercises.

(1) Suppose $z = f(x, y)$, where $x = g(s, t)$, $y = h(s, t)$, $g(1, 2) = 3$, $g_s(1, 2) = -1$, $g_t(1, 2) = 4$, $h(1, 2) = 6$, $h_s(1, 2) = -5$, $h_t(1, 2) = 10$, $f_x(3, 6) = 7$, and $f_y(3, 6) = 8$. Find $\partial z/\partial s$ and $\partial z/\partial t$ when $s = 1$ and $t = 2$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (f_x(x,y))(g_s(s,t)) + (f_y(x,y))(h_s(s,t))$$

when $(s,t)=(1,2)$, we get $(x,y)=(3,6)$. then $\left.\dfrac{dz}{ds}\right|_{\substack{s=1\\t=2}} = (7)(-1)+(8)(-5) = \boxed{-47}$

(2) If $z = x/y$, $x = se^t$ and $y = 1 + se^{-t}$, compute $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

$\boxed{\left.\dfrac{\partial z}{\partial t}\right|_{\substack{s=1\\t=2}} = 108}$ $\boxed{\text{todo}}$

Implicit Differentiation

assume $z = f(x,y)$

Ex1. Find $\frac{\partial z}{\partial x}$ if $x^3 + y^3 + z^3 = 1 - 6xyz$.

$$\frac{\partial}{\partial x}(x^3+y^3+z^3) = \frac{\partial}{\partial x}(1-6xyz)$$

$$3x^2 + 0 + 3z^2 \cdot \frac{\partial z}{\partial x} = 0 - 6yz - 6xy\frac{\partial z}{\partial x}$$

$$3z^2 \frac{\partial z}{\partial x} + 6xy\frac{\partial z}{\partial x} = -6yz - 3x^2$$

$$\frac{\partial z}{\partial x} = \frac{-6yz - 3x^2}{3z^2 + 6xy}$$

Ex2. Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Compute F_x and F_z. What is $-\frac{F_x}{F_z}$?

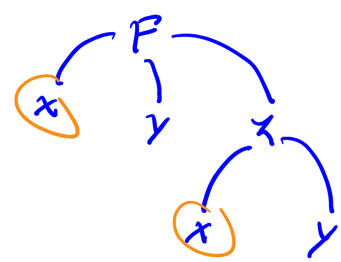
$F_x = 3x^2 + 0 + 0 + 6yz - 0$; $F_z = 0 + 0 + 3z^2 + 6xy - 0$

then $$-\frac{F_x}{F_z} = \frac{-(3x^2+6yz)}{3z^2+6xy} = \frac{\partial z}{\partial x}$$

Theorem If $F(x, y, z) = 0$ and z is a function that depends of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial}{\partial x}\big(F(x,y,z)\big) = \frac{\partial}{\partial x}(0)$$



$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$F_x(1) + F_y(0) + F_z \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

$F_z \neq 0$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \qquad \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}$$

Ex3. Find $\frac{\partial y}{\partial z}$ using the above theorem $yz + x \ln y = z^3$.

Define $F(x,y,z) = yz + x\ln(y) - z^3$

$$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y} = \frac{-(y-3z^2)}{z+\frac{x}{y}} = \frac{-y+3z^2}{z+\frac{x}{y}}$$

Sec 14.6: Directional Derivatives and the Gradient Vector

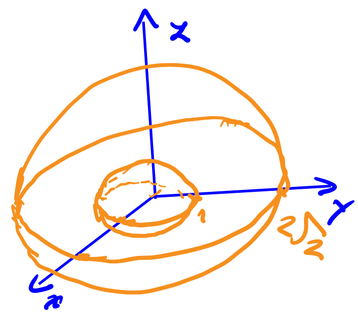
Level Surface

Let $w = g(x, y, z)$ a function in 3 variables. A level surface for $g(x, y, z)$ is the set of points in 3-d such that $g(x, y, z) = k$, for k a constant.

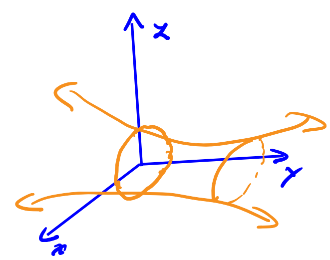
Ex1. Let $g(x, y, z) = x^2 + y^2 + z^2$. Sketch some level surfaces.

when $k=1$: $x^2+y^2+z^2=1$

when $k=8$: $x^2+y^2+z^2=8$



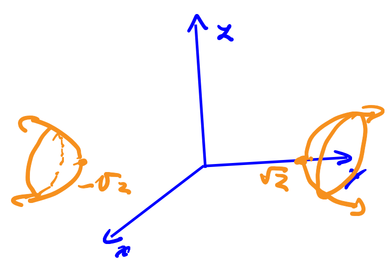
Ex2. Let $w = x^2 - y^2 + z^2$. Sketch the level surfaces $w = 1$ and $w = -2$.



when $w=1$: $x^2-y^2+z^2=1$

when $w=-2$: $x^2-y^2+z^2=-2$

$$-\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=1$$



The Gradient Vector

If f is a **function of two variables** x and y, then the **gradient of** f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle.$$

If f is a **function of three variables** x, y and z, then the **gradient of** f is the vector function ∇f defined by

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle.$$

Tangent Plane to a Level Surface

Let S be a level surface defined by $g(x, y, z) = k$, where k is a constant and let $P_0 = (x_0, y_0, z_0)$ be a point on S. The tangent plane to S at P_0 is the plane that passes through P_0 and whose normal vector is parallel to $\nabla g(P_0)$.

Ex3. Find the eq. of the tangent plane to the surface $x - z = 4\tan^{-1}(yz)$ at $(1 + \pi, 1, 1)$.

•) Define $g(x,y,z) = x - z - 4\tan^{-1}(yz)$
with level surface $g(x,y,z)=0$

•) Check that $(1+\pi, 1, 1)$ is on the level surface [that is, $g(1+\pi,1,1)=0$]

•) A vector perpendicular to the tangent plane is $\nabla g(1+\pi,1,1)$.

Finding $\nabla g(x,y,z) = \langle g_x, g_y, g_z \rangle = \left\langle 1, -4\cdot\frac{(1)(z)}{1+(yz)^2}, -1-\frac{4\cdot(1)(y)}{1+(yz)^2} \right\rangle$

then $\nabla g(1+\pi,1,1) = \left\langle 1, \frac{-4(1)(1)}{1+(1)^2}, -1-\frac{(4)(1)(1)}{1+(1)^2} \right\rangle = \langle 1, -2, -3 \rangle$

•) Eq. tangent plane at the point $(1+\pi, 1, 1)$ is

$\langle x-1-\pi, y-1, z-1 \rangle \cdot \langle 1, -2, -3 \rangle = 0$

$1(x-1-\pi) + (-2)(y-1) + (-3)(z-1) = 0$

$\boxed{x - 2y - 3z = \pi - 4}$

$g(x,y,z)=K$

DEF: The normal line to the surface at $P_0(x_0, y_0, z_0)$ is the line through P_0 parallel to $\nabla g(x_0, y_0, z_0)$.

Ex4. Find the parametric equations of the normal line to the surface $x - z = 4\tan^{-1}(yz)$ at $(1 + \pi, 1, 1)$.

Let $P_0 = (1+\pi, 1, 1)$. From Ex3, $\nabla g(P_0) = \langle 1, -2, -3 \rangle$

Parametric equations of the NORMAL line at P_0

$$\begin{cases} x = 1+\pi + t(1) \\ y = 1 + t(-2) \\ z = 1 + t(-3) \end{cases}$$

webassign:

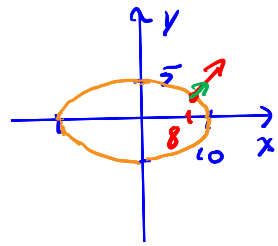
$r(t) = \langle 1+\pi+t, 1-2t, 1-3t \rangle$ "in vector form"

$(x(t), y(t), z(t)) = (\quad , \quad)$

Ex5. Consider the function $g(x,y) = \dfrac{x^2}{100} + \dfrac{y^2}{25}$ and a level curve $g(x,y) = 1$.

Sketch the level curve $g(x,y) = 1$, verify that the point $(8,3)$ is on the level curve, and sketch the vector $\nabla g(8,3)$.

•) sketch $g(x,y) = 1 \Rightarrow \dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$



•) Check that $(8,3)$ is on the level curve:

$$g(8,3) = \frac{8^2}{100} + \frac{3^2}{25} = \frac{64}{100} + \frac{9}{25} = \frac{64+36}{100} = 1$$

•) $\nabla g(x,y) = \langle g_x, g_y \rangle = \langle \frac{x}{50}, \frac{2y}{25} \rangle$

$$\nabla g(8,3) = \left\langle \frac{8}{50}, \frac{2(3)}{25} \right\rangle = \left\langle \frac{4}{25}, \frac{6}{25} \right\rangle = \frac{2}{25}\langle 2, 3 \rangle$$

To-do:

1) Find the parametric equations of the normal line to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

at the point $(-2, 1, -3)$.

Define $g(x,y,z) = \frac{x^2}{4} + y + \frac{z^2}{9}$

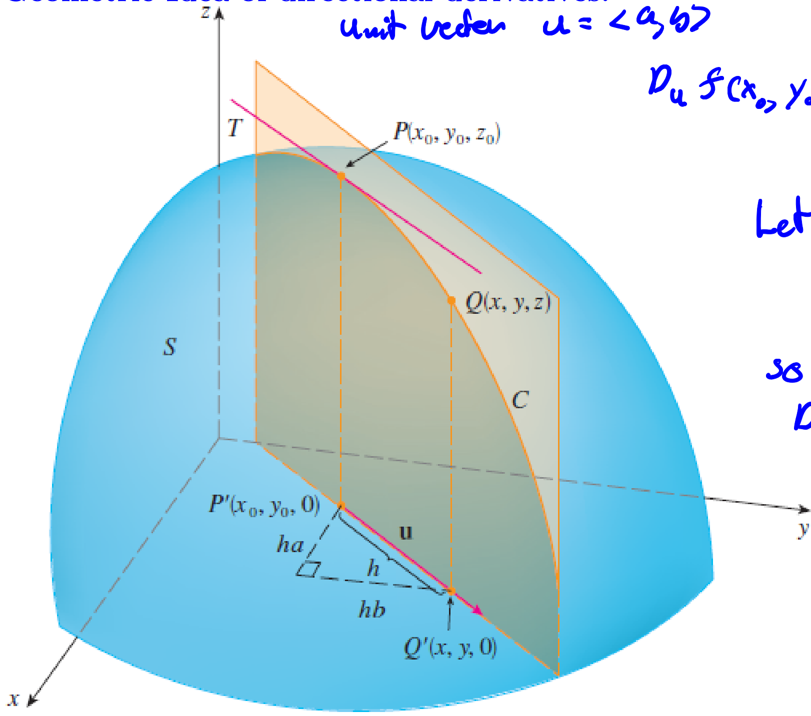
level surface: $g(x,y,z) = 3$

2) Find the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

at the point $(-2, 1, -3)$.

Geometric Idea of directional derivatives.



unit vector $u = \langle a, b \rangle$

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Let $g(h) = f(x_0 + ha, y_0 + hb)$

$g(0) = f(x_0, y_0)$

so

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= g'(0)$$

$$= \frac{d}{dh}(g(h)) \Big|_{h=0}$$

DEF: Let $z = f(x, y)$ and let (x_0, y_0) be a point in the domain of f. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is defined by

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{dh}\left\{ \underline{f\big(x_0 + ha, y_0 + hb\big)} \right\}\Big|_{h=0}$$

if this exists.

Ex1. Let $f(x, y) = x^2 y$

Find the directional derivative of f at the point $(\overset{x_0}{1}, \overset{y_0}{2})$ in the direction of the vector $\langle 5, 12 \rangle$ using the definition.

unit vector: $u = \langle \frac{5}{13}, \frac{12}{13} \rangle = \langle a, b \rangle$

$$D_u f(1, 2) = \frac{d}{dh}\left\{ f\left(1 + \frac{5h}{13}, 2 + \frac{12h}{13}\right) \right\}\Big|_{h=0}$$

$$= \frac{d}{dh}\left\{ \left(1 + \frac{5h}{13}\right)^2 \left(2 + \frac{12h}{13}\right) \right\}\Big|_{h=0}$$

$$= 2\left(1 + \frac{5h}{13}\right)\left(\frac{5}{13}\right)\left(2 + \frac{12h}{13}\right) + \left(1 + \frac{5h}{13}\right)^2 \cdot \left(\frac{12}{13}\right)\Big\}\Big|_{h=0}$$

$$= 2(1)\left(\frac{5}{13}\right)(2) + (1)^2\left(\frac{12}{13}\right) = \frac{32}{13}.$$

Theorem: Let $\mathbf{u}$ be a unit vector. If (x_0, y_0) is a point in the domain of $z = f(x, y)$, then

Dot product

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

i.e., $D_{\mathbf{u}} f(x_0, y_0)$ is the dot product of the gradient $\nabla f(x_0, y_0)$ and the unit vector $\mathbf{u}$.

Proof.

From Definition

$$D_u f(x_0, y_0) = \frac{d}{dh}\left\{ f(x_0 + ha, y_0 + hb) \right\}\Big|_{h=0}$$

$u = \langle a, b \rangle$

$x = x_0 + ha$

$y = y_0 + hb$

when $h = 0$

$x = x_0, \; y = y_0$

$$= \left\{ f_x(x,y) \cdot \frac{dx}{dh} + f_y(x,y) \cdot \frac{dy}{dh} \right\}\Big|_{h=0}$$

$$= f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b$$

$$= \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle a, b \rangle$$

$$= \nabla f(x_0, y_0) \cdot u.$$

Ex2. Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $3\vec{i} + 4\vec{j}$.

Find $D_u f(2,-1)$ where $u = \langle \frac{3}{5}, \frac{4}{5} \rangle$ "unit vector"

let's find $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2xy^3, 3x^2y^2 - 4 \rangle$

then $\nabla f(2,-1) = \langle -4, 8 \rangle$

so $D_u f(2,-1) = \nabla f(2,-1) \cdot u = \langle -4, 8 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{-12}{5} + \frac{32}{5} = \boxed{4}$

Ex3. Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $-4\vec{i} + 8\vec{j}$.

Find $D_u f(2,-1)$ where $u = \langle \frac{-4}{\sqrt{80}}, \frac{8}{\sqrt{80}} \rangle$ "unit vector"

From Ex 2: $\nabla f(2,-1) = \langle -4, 8 \rangle$

so $D_u f(2,-1) = \nabla f(2,-1) \cdot \langle \frac{-4}{\sqrt{80}}, \frac{8}{\sqrt{80}} \rangle$

$= \langle -4, 8 \rangle \cdot \langle \frac{-4}{\sqrt{80}}, \frac{8}{\sqrt{80}} \rangle = \frac{16 + 64}{\sqrt{80}} = \frac{80}{\sqrt{80}} \cdot \frac{\sqrt{80}}{\sqrt{80}} = \boxed{\sqrt{80}}$

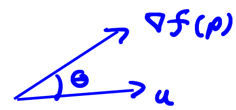
Maximizing the Directional Derivative

Suppose we have a function f of two(or three variables) and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions. We can ask the following questions: In which of these directions does f changes fastest and what is the maximum rate of change? The answers are provided by the following theorem.

Theorem. Let P be a point in the domain of the function f. Then:

1. The function f increases most rapidly in the direction of the gradient vector $\nabla f(P)$ and the maximum rate of change is $||\nabla f(P)||$.

2. The function f decreases most rapidly in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(P)$, and the minimum rate of change is $-||\nabla f(P)||$.



1) $D_u f(P) = \nabla f(p) \cdot u = ||\nabla f(p)|| \; \underbrace{||u||}_{1} \; \underbrace{\cos\theta}_{1}$

max rate of change occurs when $\theta = 0$ $[\cos\theta = 1]$

so $\nabla f(p)$ and u follow the same direction.

then, the direction in which the function increases fastest is $\nabla f(p)$ and the max rate of change is $||\nabla f(p)||$.

Ex4. Find the maximum rate of change of $f(x, y) = x^2 - x\mathrm{e}^{2y}$ at the point $(2, 0)$.

Goal: $||\nabla f(p)|| = || \langle f_x(p), f_y(p) \rangle ||$ where $p = (2,0)$.

•) $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x - e^{2y}, -2x e^{2y} \rangle$

$\Rightarrow \nabla f(2,0) = \langle 3, -4 \rangle \Rightarrow ||\nabla f(2,0)|| = \sqrt{9+16} = 5$

so, the maximum rate of change of f at $(2,0)$ is 5.

Ex5. Find the direction in which the function $f(x, y) = x^4 y - x^2 y^3$ decreases fastest at the point $(2, -3)$.

Goal: $-\nabla f(p) = -\langle f_x(p), f_y(p) \rangle$ where $p = (2,-3)$

•) $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 4x^3 y - 2xy^3, x^4 - 3x^2 y^2 \rangle$

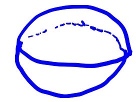
$\Rightarrow \nabla f(2,-3) = \langle 12, -92 \rangle$

so, the direction in which the function decreases fastest is

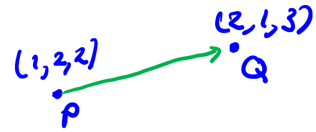
$-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$

Ex6. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$. Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

$$T(x,y,z) = \frac{K}{\sqrt{x^2+y^2+z^2}}$$



•) $T(1,2,2) = 120° \Rightarrow \dfrac{K}{\sqrt{1+4+4}} = 120° \Rightarrow K = 360°$



$\vec{PQ} = \langle 1,-1,1\rangle$

•) Goal: $\underbrace{D_v T(1,2,2)}_{\nabla T(1,2,2)\cdot v}$ where $v = \frac{1}{\sqrt{3}}\langle 1,-1,1\rangle$.

$$\nabla T(x,y,z) = \langle T_x, T_y, T_z\rangle = \left\langle \frac{-Kx}{(\sqrt{x^2+y^2+z^2})^3}, \frac{-Ky}{(\sqrt{x^2+y^2+z^2})^3}, \frac{-Kz}{(\sqrt{x^2+y^2+z^2})^3}\right\rangle.$$

Then

$$\nabla T(1,2,2) = \left\langle \frac{-K(1)}{27}, \frac{-K(2)}{27}, \frac{-K(2)}{27}\right\rangle = \frac{-K}{27}\langle 1,2,2\rangle.$$

so

$$D_v T(1,2,2) = \nabla T(1,2,2)\cdot v$$

$$= \left(\frac{-K}{27}\langle 1,2,2\rangle\right)\cdot\left(\frac{1}{\sqrt{3}}\langle 1,-1,1\rangle\right)$$

$$= \frac{-K}{27\sqrt{3}}(1-2+2) = \frac{-K}{27\sqrt{3}} = \frac{-360°}{27\sqrt{3}} = \boxed{\frac{-40°}{3\sqrt{3}}}$$

Important Remark: If $z = f(x, y)$ and $\mathbf{u} = \langle a, b\rangle$ is a unit vector, then

$$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_o), f_y(x_0, y_0)\rangle \cdot \langle a, b\rangle = af_x(x_0, y_0) + bf_y(x_0, y_0)$$

In particular, $D_{\vec{i}} f(x_0, y_0) = f_x(x_0, y_0)$ and $D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)$.

Exercise: Textbook page 1006 ♯41